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On Solving Non-homogeneous Binary Higher Degree Diophantine Equation

$$x^2 - 5xy^{2s+1} = y^{4k+3} - 5y^{4k+2}$$

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Abstract:

The non-homogeneous binary higher degree Diophantine equation given by $x^2 - 5xy^{2s+1} = y^{4k+3} - 5y^{4k+2}$ is analyzed for its patterns of non-zero distinct integral solutions.. A few interesting relations among the solutions are presented.

Keywords**:** Binary higher degree equation ,Non- Homogeneous equation , Integral solutions

Notations

$$t_{3,n} = \frac{n(n+1)}{2}$$

$$P_n^5 = \frac{n^2(n+1)}{2}$$

$$P_n^3 = \frac{n(n+1)(n+2)}{6}$$

$$S_n = 6n(n+1)+1$$

$$Th_n = 3*2^n - 1$$

Introduction:

The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-22] for quintic and cubic equations with multiple variables. This communication concerns with an interesting non-homogeneous binary higher degree equation $x^2 - 5xy^{2s+1} = y^{4k+3} - 5y^{4k+2}$ for determining its infinitely many non-zero integral points. A few interesting relations among the solutions are presented.

Method of analysis:

The given non-homogeneous binary higher degree Diophantine equation is

$$x^2 - 5x\ y^{2k+1} = y^{4k+3} - 5y^{4k+2} \qquad (1)$$

Treating (1) as quadratic in x and solving for the same , one has

$$x = \frac{y^{2k+1}[5 \pm \sqrt{4y+5}\,]}{2} \qquad (2)$$

Consider the positive sign before the square-root in (2). Choosing

$$y = y(s) = s^2 + s - 1 \qquad (3)$$

in (2) and performing some algebra , we have

$$x = x(k,s) = (s+3)\ (s^2 + s - 1)^{2k+1} \qquad (4)$$

Thus , (3) and (4) satisfy (1) . A few numerical solutions to (1) are presented in Table-1 below:

Table-1:Numerical solutions

s	**y=y(s)**	**x=x(k,s)**
1	1	4
2	5	$5*5^{2k+1}$
3	11	$6*11^{2k+1}$
4	19	$7*19^{2k+1}$
5	29	$8*29^{2k+1}$
6	41	$9*41^{2k+1}$
7	55	$10*55^{2k+1}$

Some observations between the solutions are presented below :

1. $y(s+2) - 2y(s+1) + y(s) = 2, s = 1,2,3,...$
2. $6[y(s+1) - y(s)]$ represents area of Pythagorean triangle when $s = 2\alpha^2 - 1$.
3. $y(s+2) - y(s+1)$ is a perfect square when $s = 2\alpha^2 + 4\alpha$.
4. $y(s+2) + y(s) = 4t_{3,s+1}$
5. $[y(s)]^{2k+1}(y(s+1) + y(s) + 2s) = 2s\,x(k,s)$
6. $[y(s)]^{2k+1}(y(s+1) - y(s) + 4) = 2\ x(k,s)$
7. $[y(s)]^{2k+1}(y(s+2) - y(s+1) + 2) = 2\ x(k,s)$
8. $[y(s)]^{2k+1}(y(s+2) - y(s) + 6) = 4\ x(k,s)$

9. $\frac{x(k,s)}{[y(s)]^{2k+1}} = Th_n$ when $s = 3*2^n - 4$
10. $\frac{x(k,s)}{[y(s)]^{2k+1}} = S_n$ when $s = 6n(n+1) - 2$
11. $\frac{x(k,s)}{[y(s)]^{2k+1}} = 2t_{3,n}$ when $s = n(n+1) - 3$

Remark 1

Taking the negative sign before the square-root in (2) and repeating the process as above , the corresponding integer solutions to (1) are given below:

$$y = y(s) = s^2 + s - 1 \ , \ x = x(k,s) = (2-s)\,[s^2 + s - 1]^{2k+1}$$

Note 1

To remove the square-root in (2) , let

$$\alpha^2 = 4y + 5 \tag{5}$$

which , after some algebra , is satisfied by

$$y_0 = s^2 + s - 1 \, , \alpha_0 = 2s + 1 \tag{6}$$

Assume the second solution to (5) as

$$\alpha_1 = h - \alpha_0 \, , y_1 = h + y_0 \tag{7}$$

where h is an unknown to be determined. Substituting (7) in (5) and simplifying , we have

$$h = 2\alpha_0 + 4$$

and in view of (7) ,it is seen that

$$\alpha_1 = \alpha_0 + 4, y_1 = y_0 + 2\alpha_0 + 4$$

The repetition of the above process leads to the general solution to (5) as

$$\alpha_n = \alpha_0 + 4n = 2s + 1 + 4n,$$

$$y_n = y_n(s) = y_0 + 2n\alpha_0 + 4n^2 = (s+2n)^2 + (s + 2n - 1) \tag{8}$$

Taking the positive sign before the square-root in (2) ,we get

$$x_n = x_n(s) = \frac{[y_n(s)]^{2k+1}\,[5 + \alpha_n]}{2}$$

$$= (2n + s + 3)[y_n(s)]^{2k+1} \tag{9}$$

Thus , the integer solutions to (1) are represented by (8) & (9).

Some relations between the solutions are shown below :

1. $y_{n+2}(s) - 2y_{n+1}(s) + y_n(s) = 8, n = 0,1,2,...$
2. $y_n(s+2) - 2y_n(s+1) + y_n(s) = 2, n = 0,1,2,...$
3. $[y_n(s)]^{2k+1}(y_n(s+1) - y_n(s) + 4) = 2x_n(k,s)$
4. $[y_n(s)]^{2k+1}(y_n(s+2) - y_n(s) + 6) = 4x_n(k,s)$
5. $[y_n(s)]^{2k+1}(y_n(s+2) - y_n(s+1) + 2) = 2x_n(k,s)$
6. $[y_n(s)]^{2k+1}(y_{n+1}(s) - y_n(s+1) + 2) = 2x_n(k,s)$

7. $y_n(s+2) = y_{n+1}(s)$
8. $x_n(s+2) = x_{n+1}(s)$
9. $\frac{x_n(k,s)}{[y_n(s)]^{2k+1}} = 6\,P_n^3 + 3 \quad \text{when } s = n^3 + 3n^2$
10. $\frac{x_n(k,s)}{[y_n(s)]^{2k+1}} = 2\,P_n^5 + 2n + 3 \quad \text{when } s = n^3 + n^2$

Remark 2

Taking the negative sign before the square-root in (2) ,we get

$$x_n = x_n(s) = \frac{[y_n(s)]^{2k+1}\,[5-\alpha_n]}{2}$$

$$= (2 - 2n - s)[y_n(s)]^{2k+1} \qquad (10)$$

Thus , the integer solutions to (1) are represented by (8) & (10).

Conclusion:

In this paper, we have made an attempt to obtain all integer solutions to (1). To conclude, one may search for integer solutions to other choices of homogeneous or non-homogeneous higher degree equations with multiple variables.

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